Naval Research Laboratory

Washington, DC 20375-5000

NRL Report 9146

Reduction Formulae for Kampé de Fériet Functions $F^{p:2;1}_{q:1;0}$

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Engineering Services Division

March 23, 1989

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

Form Approved
OMB No 0704-0188

1a REPORT SECURITY CLASSIFICATION: UNCLASSIFIED	1b RESTRICTIVE MARKINGS
2a SECURITY CLASSIFICATION AUTHORITY	3 DISTRIBUTION/AVAILABILITY OF REPORT: Approved for public release; distribution unlimited.
2b DECLASSIFICATION/DOWNGRADING SCHEDULE	
4 PERFORMING ORGANIZATION REPORT NUMBER(S): NRL Report 9146	5 MONITORING ORGANIZATION REPORT NUMBER(S)
6a NAME OF PERFORMING ORGANIZATION: Naval Research Laboratory; 6b OFFICE SYMBOL (If applicable): Code 2330	7a. NAME OF MONITORING ORGANIZATION
6c. ADDRESS (City, State, and ZIP Code): Washington, DC 20375-5000	7b ADDRESS (City, State, and ZIP Code)
8a. NAME OF FUNDING/SPONSORING ORGANIZATION; 8b OFFICE SYMBOL (If applicable)	9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
8c. ADDRESS (City, State, and ZIP Code)	10 SOURCE OF FUNDING NUMBERS: PROGRAM ELEMENT NO; PROJECT NO.; TASK NO; WORK UNIT ACCESSION NO

11 TITLE (Include Security Classification)
Reduction Formulae for Kampé de Fériet Functions $F^{p:2;1}_{q:1;0}$

12. PERSONAL AUTHOR(S)
Miller, A.R.

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15 PAGE COUNT
Final	FROM ______ TO ______	1989 March 23	15

16 SUPPLEMENTARY NOTATION

17 COSATI CODES			18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Kampé de Fériet functions Hypergeometric functions

19 ABSTRACT (Continue on reverse if necessary and identify by block number)

Several reduction formulae for Kampé de Fériet functions $F^{p:2;1}_{q:1;0}$ are given in terms of a finite number of generalized hypergeometric functions.

20 DISTRIBUTION/AVAILABILITY OF ABSTRACT: ☐ UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT ☐ DTIC USERS	21 ABSTRACT SECURITY CLASSIFICATION: UNCLASSIFIED
22a NAME OF RESPONSIBLE INDIVIDUAL: Allen R. Miller	22b TELEPHONE (Include Area Code): (202) 767-2215; 22c OFFICE SYMBOL: Code 2330

DD Form 1473, JUN 86 Previous editions are obsolete. SECURITY CLASSIFICATION OF THIS PAGE

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REDUCTION FORMULAE FOR KAMPE DE FERIET FUNCTIONS $F^{p:2;1}_{q:1;0}$

INTRODUCTION

In 1921, Kampé de Fériet defined and studied the functions named after him. Using the notation introduced in 1976 by Srivastava and Panda, the Kampé de Fériet functions in two variables are defined:

$$F^{p:q;k}_{l:m;n}\left[\begin{matrix}(a_p):(b_q);(c_k);\\(\alpha_l):(\beta_m);(\gamma_n);\end{matrix}\ x,y\right] := \sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}.$$

The exact region of convergence for these functions is determined by using Horn's theorem for double series. Many properties and applications of Kampé de Fériet functions are found in Refs. 1-3.

There appears, however, to be a paucity of reduction formulae for Kampé de Fériet functions. Srivastava and Karlsson [1] list only 20 nontrivial instances in which Kampé de Fériet functions can be expressed in terms of generalized hypergeometric functions. In particular, for the functions $F^{p:2;1}_{q:1;0}$, they give the three results:

$$F^{p:2;1}_{q:1;0}\left[\begin{matrix}\alpha_1,\dots,\alpha_p: & \lambda,\mu; & \nu-\lambda-\mu\ ;\\ \beta_1,\dots,\beta_q: & \nu; & \text{———};\end{matrix}\ x,x\right]$$

$$= {}_{p+2}F_{q+1}\left[\begin{matrix}\alpha_1,\dots,\alpha_p,\nu-\lambda,\nu-\mu;\\ \beta_1,\dots,\beta_q,\nu;\end{matrix}\ x\right], \tag{1}$$

$$F^{p:2;1}_{q:1;0}\left[\begin{matrix}\alpha_1,\dots,\alpha_p: & \lambda,\mu; & -1;\\ \beta_1,\dots,\beta_q: & \lambda-\mu+1; & \text{—};\end{matrix}\ x,x\right]$$

$$= {}_{p+3}F_{q+2}\left[\begin{matrix}\alpha_1,\dots,\alpha_p,\lambda-1,\mu-1,\frac{1}{2}\lambda+\frac{1}{2};\\ \beta_1,\dots,\beta_q,\lambda-\mu+1,\frac{1}{2}\lambda-\frac{1}{2};\end{matrix}\ x\right], \tag{2}$$

Manuscript approved November 2, 1988.

$$F^{p:1;1}_{q:0;0}\left[\begin{matrix}\alpha_1,\ldots,\alpha_p: & \mu; & \nu; \\ \beta_1,\ldots,\beta_q: & -; & -;\end{matrix}\; x, x\right]$$

$$= {}_{p+1}F_q\,[\alpha_1,\ldots,\alpha_p,\mu+\nu;\beta_1,\ldots,\beta_q;x], \tag{3}$$

the latter being viewed as a special case of $F^{p:2;1}_{q:1;0}$.

In a series of recent papers [4-7], the author has shown that the function $F^{0:2;1}_{2:1;0}$ is intimately connected with representations for incomplete Lipschitz-Hankel integrals of cylindrical functions (see Eqs. (14-16)), where the class of cylindrical functions C includes Bessel functions of the first kind J, modified Bessel functions I, Bessel functions of the second kind or Neumann functions Y (or N), Bessel functions of imaginary argument or MacDonald functions K, and Bessel functions of the third kind that include Hankel functions of the first and second kind, $H^{(1)}$ and $H^{(2)}$.

As a byproduct of these investigations, several reduction formulae for $F^{0:2;1}_{2:1;0}$ not included in Eqs. (1) and (2) were derived. It is the purpose of this report, in addition to collecting these results, to generalize them somewhat and to derive four reduction formulae for $F^{0:2;1}_{2:1;0}$ that were not previously given.

REDUCTION FORMULAE FOR $F^{p:2;1}_{q:1;0}\left[\begin{matrix}\mu_1,\ldots,\mu_p: & \alpha,1; & 1; \\ \nu_1,\ldots,\nu_q: & \beta; & -;\end{matrix}\; x, x\right]$

The following is derived in [4]:

$$F^{0:2;1}_{2:1;0}\left[\begin{matrix}-: & \alpha,\ 1; & 1; \\ \gamma,\delta: & \beta; & -;\end{matrix}\; x, x\right]$$

$$= \frac{1-\beta}{\alpha-\beta+1}\,{}_1F_2[1;\gamma,\delta;x] + \frac{\alpha}{\alpha-\beta+1}\,{}_2F_3[1,\alpha+1;\gamma,\delta,\beta;x]. \tag{4}$$

In particular, the following are special cases of Eq. (4) given essentially in terms of modified Bessel functions [5]:

$$F^{0:2;1}_{2:1;0}\left[\begin{matrix}-: & 2+\nu,\ 1; & 1; \\ 3+\nu,\ 3: & 5/2; & -;\end{matrix}\; \frac{x^2}{4}, \frac{x^2}{4}\right]$$

$$= \frac{2+\nu}{1+2\nu}\,\frac{48}{x^4}\left\{\cosh x - 2\,\Gamma(2+\nu)\left(\frac{2}{x}\right)^{\nu} I_\nu(x) + 1 + 2\nu\right\}, \tag{5}$$

$$F^{0:2;1}_{2:1;0}\left[\begin{matrix}\text{———}: 1+\nu,\ 1;\ \ 1; \\ 2+\nu,\ 2:\qquad 3/2;\ -;\end{matrix}\ \frac{x^2}{4},\frac{x^2}{4}\right]$$

$$= \frac{1+\nu}{1+2\nu}\,\frac{4}{x^2}\left\{\cosh x - \left(\frac{2}{x}\right)^{\nu}\Gamma(1+\nu)I_\nu(x)\right\}, \tag{6}$$

$$F^{0:2;1}_{2:1;0}\left[\begin{matrix}\text{———}: 1+\nu,\ 1;\ \ 1; \\ 1+\nu,\ 2:\qquad 3/2;\ -;\end{matrix}\ \frac{x^2}{4},\frac{x^2}{4}\right]$$

$$= \frac{2}{1+2\nu}\,\frac{1}{x}\left\{\frac{2\nu\cosh x}{x} + \sinh x - \left(\frac{2}{x}\right)^{\nu}\Gamma(1+\nu)I_{\nu-1}(x)\right\}, \tag{7}$$

$$F^{0:2;1}_{2:1;0}\left[\begin{matrix}\text{———}: 2+\nu,\ 1;\ \ 1; \\ 2+\nu,\ 3:\qquad 5/2;\ -;\end{matrix}\ \frac{x^2}{4},\frac{x^2}{4}\right]$$

$$= \frac{1}{1+2\nu}\,\frac{24}{x^4}\left\{x\sinh x + 2\nu\cosh x - 4\Gamma(2+\nu)\left(\frac{2}{x}\right)^{\nu-1} I_{\nu-1}(x) + 2\nu(1+2\nu)\right\}. \tag{8}$$

Equation (4) is, in fact, included in the reduction formula

$$F^{p:2;1}_{q:1;0}\left[\begin{matrix}\mu_1,\ldots,\mu_p: \alpha,\ 1;\ \ 1; \\ \nu_1,\ldots,\nu_q:\qquad \beta;\ -;\end{matrix}\ x, x\right]$$

$$= \frac{1-\beta}{1-\beta+\alpha}\ {}_{p+1}F_q\,[\mu_1,\ldots,\mu_p,1;\nu_1,\ldots,\nu_q;x]$$

$$+ \frac{\alpha}{1-\beta+\alpha}\ {}_{p+2}F_{q+1}\,[\mu_1,\ldots,\mu_p,1,\alpha+1;\nu_1,\ldots,\nu_q,\beta;x], \tag{9}$$

which we now show.

By definition

$$F^{p:2;1}_{q:1;0}\left[\begin{matrix}\mu_1,\ldots,\mu_p: \alpha,\ 1;\ \ 1; \\ \nu_1,\ldots,\nu_q:\qquad \beta;\ -;\end{matrix}\ x, x\right]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\mu_1)_{m+n}\cdots(\mu_p)_{m+n}}{(\nu_1)_{m+n}\cdots(\nu_q)_{m+n}}\ \frac{(\alpha)_m\ (1)_m\ (1)_n}{(\beta)_m}\ \frac{x^m x^n}{m!n!}$$

$$= \sum_{r=0}^{\infty}\left[\sum_{m=0}^{r}\frac{(\alpha)_m}{(\beta)_m}\right]\frac{(\mu_1)_r\cdots(\mu_p)_r}{(\nu_1)_r\cdots(\nu_q)_r}\,x^r.$$

Since [8, Eq. (7.1.1), p. 151]

$$\sum_{m=0}^{r} \frac{(\alpha)_m}{(\beta)_m} = \frac{1}{1-\beta+\alpha} \left[1 - \beta + \alpha \frac{(\alpha+1)_r}{(\beta)_r} \right],$$

we arrive at

$$F^{p:2;1}_{q:1;0} \left[\begin{matrix} \mu_1, \ldots, \mu_p: \alpha, 1; & 1; \\ \nu_1, \ldots, \nu_q: \quad \beta; & -; \end{matrix} x, x \right]$$

$$= \frac{1-\beta}{1-\beta+\alpha} \sum_{r=0}^{\infty} \frac{(\mu_1)_r \ldots (\mu_p)_r \, (1)_r}{(\nu_1)_r \ldots (\nu_q)_r} \frac{x^r}{r!}$$

$$+ \frac{\alpha}{1-\beta+\alpha} \sum_{r=0}^{\infty} \frac{(\mu_1)_r \ldots (\mu_p)_r}{(\nu_1)_r \ldots (\nu_q)_r} \frac{(1)_r \, (\alpha+1)_r}{(\beta)_r} \frac{x^r}{r!},$$

and Eq. (9) follows.

For brevity, we define below two functions that occur often in what follows:

$${}_p\boldsymbol{S}_q \, [(a_p); (b_q); z]$$

$$:= \frac{1}{2} \{ e^z \, {}_pF_q [(a_p); (b_q); -2z] - e^{-z} \, {}_pF_q \, [(a_p); (b_q); 2z] \},$$

$${}_p\boldsymbol{C}_q \, [(a_p); (b_q); z]$$

$$:= \frac{1}{2} \{ e^z \, {}_pF_q \, [(a_p); (b_q); -2z] + e^{-z} \, {}_pF_q \, [(a_p); (b_q); 2z] \}.$$

We observe that

$${}_p\boldsymbol{S}_q \, [(a_p); (b_q); z] = \left\{ 1 - 2 \frac{\prod_{k=1}^{p} a_k}{\prod_{k=1}^{q} b_k} \right\} z + O(z^3) \qquad z \to 0,$$

$${}_p\boldsymbol{C}_q \, [(a_p); (b_q); z] = 1 + O(z^2) \qquad z \to 0.$$

Hence,

$$\lim_{z \to 0} \frac{{}_pS_q\ [(a_p);\ (b_q);\ z]}{z} = 1 - 2\,\frac{\prod_{k=1}^{p} a_k}{\prod_{k=1}^{q} b_k},$$

$${}_pC_q\ [(a_p);\ (b_q);\ 0] = 1.$$

REDUCTION FORMULAE FOR $F^{0:2;1}_{2:1;0}$

In addition, using the functions ${}_2C_2$ and ${}_2S_2$, we have from results in [6]

$$F^{0:2:1}_{2:1:0}\left[\begin{matrix} \overline{\qquad\qquad} & : \frac{1+\mu+\nu}{2}, \frac{2+\mu+\nu}{2} & ; 1; \\ \frac{2+\mu+\nu}{2}, \frac{3+\mu+\nu}{2} & : \quad 1+\nu & ; -; \end{matrix}\ \frac{x^2}{4}, \frac{x^2}{4}\right]$$

$$= {}_2C_2\left[\begin{matrix} 1+\mu+\nu,\ 1/2+\nu; \\ 2+\mu+\nu,\ \ 1+2\nu; \end{matrix}\ x\right], \tag{10}$$

$$F^{0:2;1}_{2:1:0}\left[\begin{matrix} \overline{\qquad\qquad} & : \frac{1+\mu+\nu}{2}, \frac{2+\mu+\nu}{2} & ; 1; \\ \frac{3+\mu+\nu}{2}, \frac{4+\mu+\nu}{2} & : \quad 1+\nu & ; -; \end{matrix}\ \frac{x^2}{4}, \frac{x^2}{4}\right]$$

$$= \frac{2+\mu+\nu}{2x}\ {}_2S_2\left[\begin{matrix} 1+\mu+\nu,\ 1/2+\nu; \\ 2+\mu+\nu,\ \ 1+2\nu; \end{matrix}\ x\right]. \tag{11}$$

Equations (10) and (11) may also be written [6]

$$F^{0:2;1}_{2:1;0}\left[\begin{matrix} \text{———} : \frac{1+\mu+\nu}{2}, \frac{2+\mu+\nu}{2}; & 1; \\ \frac{2+\mu+\nu}{2}, \frac{3+\mu+\nu}{2} : \quad 1+\nu; & -; \end{matrix} \frac{x^2}{4}, \frac{x^2}{4}\right]$$

$$= \cosh x \; {}_3F_4\left[\begin{matrix} \frac{1+\mu+\nu}{2}, \frac{1/2+\nu}{2}, \frac{3/2+\nu}{2}; \\ \frac{3+\mu+\nu}{2}, 1/2+\nu, 1+\nu, 1/2; \end{matrix} x^2\right]$$

$$- \frac{1+\mu+\nu}{2+\mu+\nu} x \sinh x \; {}_3F_4\left[\begin{matrix} \frac{2+\mu+\nu}{2}, \frac{3/2+\nu}{2}, \frac{5/2+\nu}{2}; \\ \frac{4+\mu+\nu}{2}, 3/2+\nu, 1+\nu, 3/2; \end{matrix} x^2\right], \tag{12}$$

$$F^{0:2;1}_{2:1;0}\left[\begin{matrix} \text{———} : \frac{1+\mu+\nu}{2}, \frac{2+\mu+\nu}{2}; & 1; \\ \frac{3+\mu+\nu}{2}, \frac{4+\mu+\nu}{2} : \quad 1+\nu; & -; \end{matrix} \frac{x^2}{4}, \frac{x^2}{4}\right]$$

$$= (2+\mu+\nu)\frac{\sinh x}{x} \; {}_3F_4\left[\begin{matrix} \frac{1+\mu+\nu}{2}, \frac{1/2+\nu}{2}, \frac{3/2+\nu}{2}; \\ \frac{3+\mu+\nu}{2}, 1/2+\nu, 1+\nu, 1/2; \end{matrix} x^2\right]$$

$$- (1+\mu+\nu) \cosh x \; {}_3F_4\left[\begin{matrix} \frac{2+\mu+\nu}{2}, \frac{3/2+\nu}{2}, \frac{5/2+\nu}{2}; \\ \frac{4+\mu+\nu}{2}, 3/2+\nu, 1+\nu, 3/2; \end{matrix} x^2\right] \tag{13}$$

PRELIMINARY RESULTS AND DEFINITIONS

The general incomplete Lipschitz-Hankel integral of cylindrical functions $C_\nu(z)$ may be defined as the following function of two complex variables:

$$Ce_{\mu,\nu}(a, z) := \int_0^z e^{at} t^\mu C_\nu(t)\, dt \,. \tag{14}$$

Here the symbol e denotes the presence of the exponential function, and μ, ν may be complex. Analogously, we may define integrals that contain the functions $\sin(at)$ and $\cos(at)$ in place of $\exp(at)$:

$$C_{s_{\mu,\nu}}(a, z) := \int_0^z \sin(at)\, t^{\mu} C_{\nu}(t)\, dt, \tag{15}$$

$$C_{c_{\mu,\nu}}(a, z) := \int_0^z \cos(at)\, t^{\mu} C_{\nu}(t)\, dt. \tag{16}$$

To assure convergence of $C_{e_{\mu,\nu}}(a, z)$ and $C_{c_{\mu,\nu}}(a, z)$, it is necessary that Re $(\mu + 1) > |\text{Re}\ \nu|$ when $C = K, Y, H^{(1)}, H^{(2)}$; Re $(1 + \mu + \nu) > 0$ when $C = I, J$. For convergence of $C_{s_{\mu,\nu}}(a, z)$, replace μ by $\mu + 1$ in the latter two inequalities.

Defining

$$\xi := \begin{cases} 1: C = I, K \\ -1: C = H, J, Y \end{cases} \qquad \eta := \begin{cases} 1: C = K \\ -1: C = H, I, J, Y \end{cases}$$

$$Q\,[A_1(\mu, \nu); x, y] := F^{0:2:1}_{2:1:0}\left[\begin{matrix} \text{———} : \dfrac{\mu+\nu+1}{2}, \dfrac{\mu-\nu+1}{2}; 1; \\ \dfrac{\mu+\nu+1}{2}, \dfrac{\mu-\nu+3}{2} : \quad 1/2 \quad ; -; \end{matrix}\ x, y\right]$$

$$Q\,[A_2(\mu, \nu); x, y] := F^{0:2:1}_{2:1:0}\left[\begin{matrix} \text{———} : \dfrac{\mu+\nu+1}{2}, \dfrac{\mu-\nu+1}{2}; 1; \\ \dfrac{\mu+\nu+3}{2}, \dfrac{\mu-\nu+3}{2} : \quad 1/2 \quad ; -; \end{matrix}\ x, y\right]$$

$$Q\,[B_1(\mu, \nu); x, y] := F^{0:2:1}_{2:1:0}\left[\begin{matrix} \text{———} : \dfrac{\mu+\nu+2}{2}, \dfrac{\mu-\nu+2}{2}; 1; \\ \dfrac{\mu+\nu+2}{2}, \dfrac{\mu-\nu+4}{2} : \quad 3/2 \quad ; -; \end{matrix}\ x, y\right]$$

$$Q\,[B_2(\mu, \nu); x, y] := F^{0:2:1}_{2:1:0}\left[\begin{matrix} \text{———} : \dfrac{\mu+\nu+2}{2}, \dfrac{\mu-\nu+2}{2}; 1; \\ \dfrac{\mu+\nu+4}{2}, \dfrac{\mu-\nu+4}{2} : \quad 3/2 \quad ; -; \end{matrix}\ x, y\right],$$

it is shown in [7] that

$$C_{s_{\mu,\nu}}(a,z) = \frac{az^{2+\mu}}{\mu-\nu+2}\left\{C_{\nu}(z) Q\left[B_1; \frac{-a^2z^2}{4}, \frac{\xi z^2}{4}\right] + \frac{\eta z}{\mu+\nu+2} C_{\nu-1}(z) Q\left[B_2; \frac{-a^2z^2}{4}, \frac{\xi z^2}{4}\right]\right\}, \tag{17}$$

$$C_{c_{\mu,\nu}}(a,z) = \frac{z^{1+\mu}}{\mu-\nu+1}\left\{C_{\nu}(z) Q\left[A_1; \frac{-a^2z^2}{4}, \frac{\xi z^2}{4}\right] + \frac{\eta z}{\mu+\nu+1} C_{\nu-1}(z) Q\left[A_2; \frac{-a^2z^2}{4}, \frac{\xi z^2}{4}\right]\right\}. \tag{18}$$

ADDITIONAL RESULTS FOR $F^{0:2;1}_{2:1;0}$

In Eq (17), set firstly $\xi = 1$, $\eta = -1$, $a = i$ and secondly $\xi = 1$, $\eta = 1$, $a = i$. This gives the system of linear equations in two unknowns $Q[B_1; z^2/4, z^2/4]$ and $Q[B_2; z^2/4, z^2/4]$:

$$I_\nu(z)\, Q\left[B_1; \frac{z^2}{4}, \frac{z^2}{4}\right] - \frac{z}{\mu+\nu+2} I_{\nu-1}(z)\, Q\left[B_2; \frac{z^2}{4}, \frac{z^2}{4}\right]$$

$$= \frac{\mu-\nu+2}{z^{2+\mu}} \int_0^z \sinh t \; t^\mu I_\nu(t)\, dt, \tag{19}$$

$$K_\nu(z)\, Q\left[B_1; \frac{z^2}{4}, \frac{z^2}{4}\right] + \frac{z}{\mu+\nu+2} K_{\nu-1}(z)\, Q\left[B_2; \frac{z^2}{4}, \frac{z^2}{4}\right]$$

$$= \frac{\mu-\nu+2}{z^{2+\mu}} \int_0^z \sinh t \; t^\mu K_\nu(t)\, dt. \tag{20}$$

Now making the previous substitutions in Eq. (18) gives the system of linear equations in unknowns $Q[A_1; z^2/4, z^2/4]$ and $Q[A_2; z^2/4, z^2/4]$:

$$I_\nu(z)\, Q\left[A_1; \frac{z^2}{4}, \frac{z^2}{4}\right] - \frac{z}{\mu+\nu+1} I_{\nu-1}(z)\, Q\left[A_2; \frac{z^2}{4}, \frac{z^2}{4}\right]$$

$$= \frac{\mu-\nu+1}{z^{1+\mu}} \int_0^z \cosh t \; t^\mu I_\nu(t)\, dt, \tag{21}$$

$$K_\nu(z)\, Q\left[A_1; \frac{z^2}{4}, \frac{z^2}{4}\right] + \frac{z}{\mu+\nu+1} K_{\nu-1}(z)\, Q\left[A_2; \frac{z^2}{4}, \frac{z^2}{4}\right]$$

$$= \frac{\mu-\nu+1}{z^{1+\mu}} \int_0^z \cosh t \; t^\mu K_\nu(t)\, dt. \tag{22}$$

Solving the systems Eqs. (19) and (20), and Eqs. (21) and (22), respectively, using Cramer's rule and noting the Wronskian

$$I_\nu(z)\, K_{\nu-1}(z) + I_{\nu-1}(z)\, K_\nu(z) = 1/z, \tag{23}$$

we obtain

$$Q\left[B_1; \frac{z^2}{4}, \frac{z^2}{4}\right] = \frac{\mu-\nu+2}{z^{1+\mu}} \begin{vmatrix} \int_0^z \sinh t \; t^\mu I_\nu(t)dt & -I_{\nu-1}(z) \\ \int_0^z \sinh t \; t^\mu K_\nu(t)dt & K_{\nu-1}(z) \end{vmatrix}, \tag{24}$$

$$Q\left[B_2; \frac{z^2}{4}, \frac{z^2}{4}\right] = \frac{(\mu - \nu + 2)(\mu + \nu + 2)}{z^{2+\mu}} \begin{vmatrix} I_\nu(z) & \int_0^z \sinh t\ t^\mu I_\nu(t)dt \\ K_\nu(z) & \int_0^z \sinh t\ t^\mu K_\nu(t)dt \end{vmatrix}, \tag{25}$$

$$Q\left[A_1; \frac{z^2}{4}, \frac{z^2}{4}\right] = \frac{(\mu - \nu + 1)}{z^\mu} \begin{vmatrix} \int_0^z \cosh t\ t^\mu I_\nu(t)dt & -I_{\nu-1}(z) \\ \int_0^z \cosh t\ t^\mu K_\nu(t)dt & K_{\nu-1}(z) \end{vmatrix}, \tag{26}$$

$$Q\left[A_2; \frac{z^2}{4}, \frac{z^2}{4}\right] = \frac{(\mu - \nu + 1)(\mu + \nu + 1)}{z^{1+\mu}} \begin{vmatrix} I_\nu(z) & \int_0^z \cosh t\ t^\mu I_\nu(t)dt \\ K_\nu(z) & \int_0^z \cosh t\ t^\mu K_\nu(t)dt \end{vmatrix}. \tag{27}$$

Equations (24) and (25) are valid for $\mathrm{Re}(\mu \pm \nu + 2) > 0$; Eqs. (26) and (27) are valid for $\mathrm{Re}(\mu \pm \nu + 1) > 0$. Further, Eqs. (24-27) provide integral representations for the function Q appearing in each equation.

Using [9, p. 117, Eq. (2)], we evaluate the four integrals in Eqs. (24-27) in terms of the functions ${}_2\boldsymbol{C}_2$ and ${}_2\boldsymbol{S}_2$:

$$\int_0^z \sinh t\ t^\mu\ C_\nu(t)dt = \frac{z^{\mu+1}}{\mu + \nu + 1} \Bigg\{ \sinh z\ C_\nu(z)$$

$$- \frac{zC_\nu(z)}{\mu + \nu + 2}\ {}_2\boldsymbol{C}_2\left[\begin{matrix} 1, & \mu + 3/2; \\ \mu - \nu + 1, & \mu + \nu + 3; \end{matrix}\ z\right] + \frac{\delta z C_{\nu+1}(z)}{\mu - \nu + 1}\ {}_2\boldsymbol{S}_2\left[\begin{matrix} 1, & \mu + 3/2; \\ \mu - \nu + 2, & \mu + \nu + 2; \end{matrix}\ z\right]\Bigg\},$$

$$\int_0^z \cosh t\ t^\mu\ C_\nu(t)dt = \frac{z^{1+\mu}}{\mu + \nu + 1} \Bigg\{ \cosh z\ C_\nu(z)$$

$$- \frac{zC_\nu(z)}{\mu + \nu + 2}\ {}_2\boldsymbol{S}_2\left[\begin{matrix} 1, & \mu + 3/2; \\ \mu - \nu + 1, & \mu + \nu + 3; \end{matrix}\ z\right] + \frac{\delta z C_{\nu+1}(z)}{\mu - \nu + 1}\ {}_2\boldsymbol{C}_2\left[\begin{matrix} 1, & \mu + 3/2; \\ \mu - \nu + 2, & \mu + \nu + 2; \end{matrix}\ z\right]\Bigg\},$$

where $\delta = 1$ if $C = K$, $\delta = -1$ if $C = I$.

Using these results in Eqs. (24-27), the F-symbol for Q and taking note of Eq. (23) together with the easily proved identity

$$K_{\nu+1}(z)\, I_{\nu-1}(z) - I_{\nu+1}(z) K_{\nu-1}(z) = 2\nu/z^2,$$

we deduce the four reduction formulae:

$$F^{0:2:1}_{2:1:0}\left[\begin{array}{l} \rule{5cm}{0.4pt} : \frac{\mu+\nu+2}{2}, \frac{\mu-\nu+2}{2};\ 1; \\ \frac{\mu+\nu+2}{2}, \frac{\mu-\nu+4}{2} : \quad 3/2 \quad ;\ -; \end{array} \frac{z^2}{4}, \frac{z^2}{4}\right] \tag{28}$$

$$= \frac{\mu-\nu+2}{\mu+\nu+1}\left\{\frac{\sinh z}{z} - \frac{1}{\mu+\nu+2}\,{}_2C_2\left[\begin{array}{c}1,\ \mu+3/2; \\ \mu-\nu+1, \mu+\nu+3;\end{array} z\right] + \frac{2\nu}{\mu-\nu+1}\frac{1}{z}\,{}_2S_2\left[\begin{array}{c}1,\ \mu+3/2; \\ \mu-\nu+2, \mu+\nu+2;\end{array} z\right]\right\},$$

$$F^{0:2:1}_{2:1:0}\left[\begin{array}{l} \rule{5cm}{0.4pt} : \frac{\mu+\nu+2}{2}, \frac{\mu-\nu+2}{2};\ 1; \\ \frac{\mu+\nu+4}{2}, \frac{\mu-\nu+4}{2} : \quad 3/2 \quad ;\ -; \end{array} \frac{z^2}{4}, \frac{z^2}{4}\right]$$

$$= \frac{(\mu+\nu+2)(\mu-\nu+2)}{(\mu+\nu+1)(\mu-\nu+1)}\frac{1}{z}\,{}_2S_2\left[\begin{array}{c}1,\quad \mu+3/2; \\ \mu-\nu+2, \mu+\nu+2;\end{array} z\right], \tag{29}$$

$$F^{0:2:1}_{2:1:0}\left[\begin{array}{l} \rule{5cm}{0.4pt} : \frac{\mu+\nu+1}{2}, \frac{\mu-\nu+1}{2};\ 1; \\ \frac{\mu+\nu+1}{2}, \frac{\mu-\nu+3}{2} : \quad 1/2 \quad ;\ -; \end{array} \frac{z^2}{4}, \frac{z^2}{4}\right] \tag{30}$$

$$= \frac{\mu-\nu+1}{\mu+\nu+1}\left\{\cosh z - \frac{z}{\mu+\nu+2}\,{}_2S_2\left[\begin{array}{c}1,\ \mu+3/2; \\ \mu-\nu+1, \mu+\nu+3;\end{array} z\right] + \frac{2\nu}{\mu-\nu+1}\,{}_2C_2\left[\begin{array}{c}1,\ \mu+3/2; \\ \mu-\nu+2, \mu+\nu+2;\end{array} z\right]\right\},$$

$$F^{0:2:1}_{2:1:0}\left[\begin{array}{l} \rule{5cm}{0.4pt} : \frac{\mu+\nu+1}{2}, \frac{\mu-\nu+1}{2};\ 1; \\ \frac{\mu+\nu+3}{2}, \frac{\mu-\nu+3}{2} : \quad 1/2 \quad ;\ -; \end{array} \frac{z^2}{4}, \frac{z^2}{4}\right]$$

$$= {}_2C_2\left[\begin{array}{c}1,\quad \mu+3/2; \\ \mu-\nu+2, \mu+\nu+2;\end{array} z\right]. \tag{31}$$

If we set $\mu = \nu$, then it is easily shown that Eq. (28) reduces to Eq. (7), and Eq. (29) reduces to Eq. (6). Equations (30) and (31) reduce to $F^{0:1;1}_{2:0;0}$ functions (see Eqs. (22) and (23) of Ref. 5), whose reduction is given by Eq. (3).

We remark that Eqs. (28-31) are valid for $|z| < \infty$ and for all μ, ν such that each function exists. Hence the restrictions on the existence of the integrals in Eqs. (24-27) do not apply here.

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